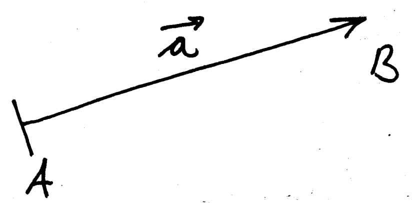
Vektori

Vektor definišemo kao orjentisanu duž.



$\vec{AB} = \vec{a}$

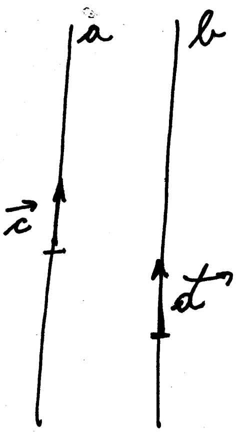
$\vec{O}$ nula vektor

Svaki vektor ima intenzitet, pravac i smijer

$|\vec{a}|$ intenzitet (veličina duži)

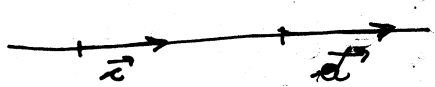
$|\vec{AB}| \geq 0 \quad \forall$ tačke A i B

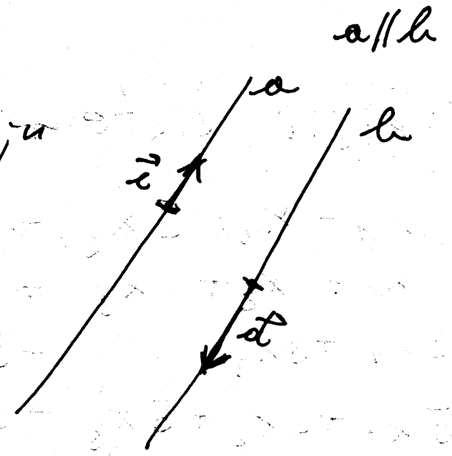
Pravac vektora određena je pravom na kojoj vektor leži i tu pravu zovemo nosač vektora.



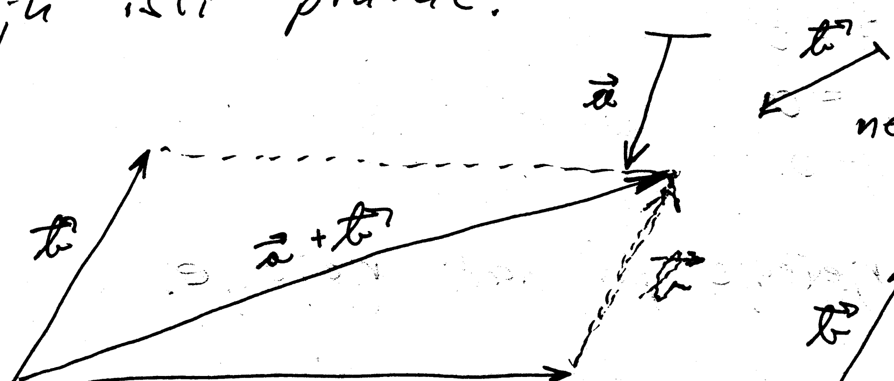
$a \parallel b$ - prave

$\vec{c}$ i $\vec{d}$ vektori imaju isti pravac

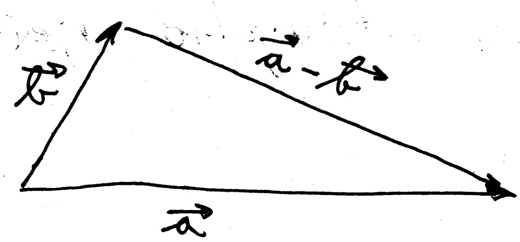




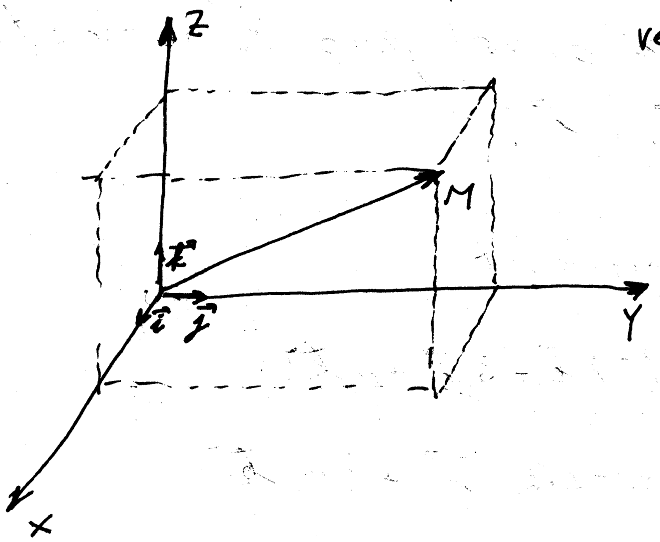
Smjer vektora određen je izborom početne i završne tačke. Vektori se mogu porediti po smjeru ako imaju isti pravac.



nemaju isti pravac



Ako je $\vec{a}$ jedinični vektor tada je $|\vec{a}|=1$.



vektor $\vec{OM}$ u koordinatnom sistemu

$$\vec{OM} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\vec{OM} = (x, y, z)$$

$M_1(x_1, y_1, z_1)$

$M_2(x_2, y_2, z_2)$

$\vec{M_1M_2}(x_2-x_1, y_2-y_1, z_2-z_1)$

$$\vec{a} = (a_1, a_2, a_3)$$

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

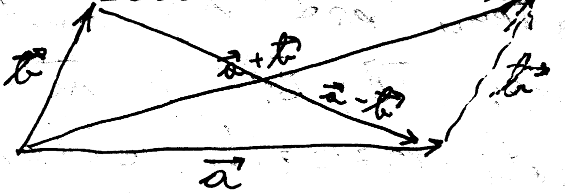
Vektori $\vec{a}, \vec{b}, \vec{c}$ su linearno $\underbrace{\text{zavisni}}_{}$ (komplanarni – nalaze se u istoj ravni) ako postoje skalari α, β i γ različiti od 0 tako da važi

$$\alpha\vec{a} + \beta\vec{b} + \gamma\vec{c} = \vec{0}$$

$\vec{a} = \lambda\vec{b} + \mu\vec{c}$ razlaganje vektora $\vec{a}$ preko vektora $\vec{b}$ i $\vec{c}$ (vektori se nalaze u istoj ravni)

1. Kakav međusobni položaj zauzimaju vektori $\vec{a}$ i $\vec{b}$ ako je $|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|$.

Rj. Pretpostavimo da su vektori $\vec{a}$ i $\vec{b}$ dovedeni na zajednički početak:



Imamo paralelogram kod koga su dijagonale jednake.

Kad je ovo moguće?

Ovo je moguće samo u slučaju kad je dati paralelogram pravougaonik ili kvadrat. I u jednom i u drugom slučaju imamo da je $\vec{a} \perp \vec{b}$ ($\vec{a}$ i $\vec{b}$ su okomiti vektori).

2. Ispitati linearnu zavisnost vektora $\vec{a}=(2,3,-4)$, $\vec{b}=(3,-2,0)$ i $\vec{c}=(0,1,1)$.

Rj: $\alpha\vec{a}+\beta\vec{b}+\gamma\vec{c}=\vec{0}$

$\alpha(2,3,-4)+\beta(3,-2,0)+\gamma(0,1,1)=(0,0,0)$

$$\begin{aligned} 2\alpha+3\beta &= 0\\ 3\alpha-2\beta+\gamma &= 0\\ -4\alpha+\gamma &= 0\end{aligned}$$

$$\det M=\begin{vmatrix}2&3&0\\3&-2&1\\-4&0&1\end{vmatrix}\overset{III_v-II_v}{=\!=\!=}\begin{vmatrix}2&3&0\\3&-2&1\\-7&2&0\end{vmatrix}=$$

$$=(-1)\begin{vmatrix}2&3\\-7&2\end{vmatrix}=(-1)(4+21)=-25$$

$\det M\neq 0$

sistem ima samo trivijalna rješenja $((0,0,0))$

Vektori $\vec{a},\vec{b}$ i $\vec{c}$ su linearno nezavisni.

3. Dokazati da su vektori $\vec{a}=(3,1,8)$, $\vec{b}=(3,4,5)$ i $\vec{c}=(2,3,3)$ linearno zavisni.

$\alpha\vec{a}+\beta\vec{b}+\gamma\vec{c}=\vec{0}$

$\alpha(3,1,8)+\beta(3,4,5)+\gamma(2,3,3)=(0,0,0)$

$$\begin{aligned} 3\alpha+3\beta+2\gamma &= 0\\ \alpha+4\beta+3\gamma &= 0\\ 8\alpha+5\beta+3\gamma &= 0\end{aligned}$$

$$\det M=\begin{vmatrix}3&3&2\\1&4&3\\8&5&3\end{vmatrix}\overset{I_v-II_v\cdot 3}{\underset{III_v-II_v\cdot 8}{=\!=\!=}}\begin{vmatrix}0&-9&-7\\1&4&3\\0&-27&-21\end{vmatrix}=(-1)\begin{vmatrix}-9&-7\\-27&-21\end{vmatrix}$$

$$=(-1)(-9)(-7)\begin{vmatrix}1&1\\3&3\end{vmatrix}=0$$

$\det M=0$

$\operatorname{rang} M<3$

Sistem ima netrivijalna rješenja

Vektori $\vec{a},\vec{b}$ i $\vec{c}$ su linearno zavisni.

4. Diskutovati linearnu zavisnost vektora $\vec{a}=(3,-8,2)$, $\vec{b}=(7,6,5)$ i $\vec{c}=(5,2,6-\lambda)$ u zavisnosti od parametra λ.

Rj. $\det M=182-74\lambda$

1° $\lambda=\frac{182}{74}$ vektori linearno zavisni

2° $\lambda\neq\frac{182}{74}$ vektori linearno nezavisni

5. Odrediti parametar λ tako da vektori $\vec{a}=\lambda\vec{i}+\vec{j}+4\vec{k}$, $\vec{b}=\vec{i}-2\lambda\vec{j}$ i $\vec{c}=3\lambda\vec{i}-3\vec{j}+4\vec{k}$ budu komplanarni pa za tako dobijeno λ razložiti vektor $\vec{a}$ preko vektora $\vec{b}$ i $\vec{c}$.

Rj. $\alpha\vec{a}+\beta\vec{b}+\gamma\vec{c}=0$ uslov komplanarnosti

$$\alpha(\lambda,1,4)+\beta(1,-2\lambda,0)+\gamma(3\lambda,-3,4)=(0,0,0)$$

$$\begin{aligned} \lambda\alpha+\beta+3\lambda\gamma&=0\\ \alpha-2\lambda\beta-3\gamma&=0\\ 4\alpha+\quad 4\gamma&=0 \end{aligned}$$

sistem, α, β i γ su nepoznate

$$D=\begin{vmatrix} \lambda & 1 & 3\lambda\\ 1 & -2\lambda & -3\\ 4 & 0 & 4\end{vmatrix} \overset{III_k-I_k}{=\!=\!=} \begin{vmatrix} \lambda & 1 & 2\lambda\\ 1 & -2\lambda & -4\\ 4 & 0 & 0\end{vmatrix}=$$

$$=4\begin{vmatrix} 1 & 2\lambda\\ -2\lambda & -4\end{vmatrix}=4\cdot 2\begin{vmatrix} 1 & 2\lambda\\ -\lambda & -2\end{vmatrix}=8\cdot 2\begin{vmatrix} 1 & \lambda\\ -\lambda & -1\end{vmatrix}$$

$$D=16(-1+\lambda^2)=16(\lambda^2-1)$$

Za $\lambda=\pm 1$ imamo da je $D=0$ $\Rightarrow$ sistem ima beskonačno mnogo rješenja (za $\lambda=\pm1$).

Za $\lambda=\pm1$ vektori $\vec{a}$, $\vec{b}$ i $\vec{c}$ su komplanarni. Uzmimo da je $\lambda=1$:

$\vec{a}=(1,1,4)$
$\vec{b}=(1,-2,0)$
$\vec{c}=(3,-3,4)$

$$\vec{a}=\alpha\vec{b}+\beta\vec{c}$$

$$\alpha(1,-2,0)+\beta(3,-3,4)=(1,1,4)$$

$$\begin{aligned} \alpha+3\beta&=1\\ -2\alpha-3\beta&=1\\ 4\beta&=4 \end{aligned}$$

$\beta=1$
$\alpha+3=1$
$\alpha=-2$

za $\lambda=1$
$\vec{a}=-2\vec{b}+\vec{c}$
razlaganje vektora $\vec{a}$ preko vektora $\vec{b}$ i $\vec{c}$

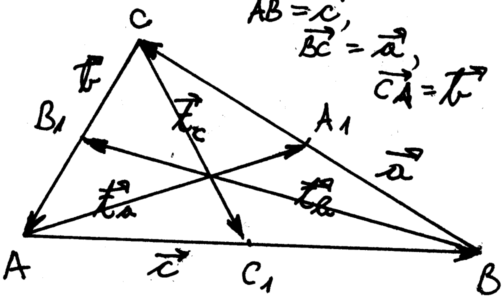
Za $\lambda=-1$ vektor $\vec{a}$ razložen preko vektora $\vec{b}$ i $\vec{c}$:

$$\vec{a}=2\vec{b}+\vec{c}$$

6. Stranice trougla su vektori $\vec{a}$, $\vec{b}$ i $\vec{c}$. Pomoću ovih vektora izraziti težišne linije trougla (vidi sliku).

Rj.

$\vec{AB}=\vec{c}$, $\vec{BC}=\vec{a}$, $\vec{CA}=\vec{b}$



Težišna linija je duž koja spaja tjeme trougla sa sredinom stranice nasprem tog tjemena.

$$\vec{t}_a=\vec{AA_1}=\vec{AB}+\vec{BA_1}=\vec{c}+\frac{1}{2}\vec{a}$$

$$\vec{t}_a=\vec{AA_1}=\vec{AC}+\vec{CA_1}=-\vec{CA}-\frac{1}{2}\vec{BC}=-\vec{b}-\frac{1}{2}\vec{a}$$

Za vježbu: $\vec{t}_b=\vec{a}+\frac{1}{2}\vec{b}=-\vec{c}-\frac{1}{2}\vec{b}$, $\vec{t}_c=\vec{b}-\vec{c}=-\vec{a}-\frac{1}{2}\vec{c}$

7. Data su tjemena paralelograma $\square ABCD$

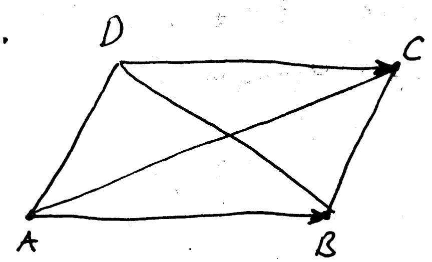
$A(-3, 2, \lambda)$, $B(3, -3, 1)$ i $C(5, \lambda, 2)$.

a) Odrediti tjeme D

b) Odrediti λ tako da je $|\overrightarrow{AD}| = \sqrt{14}$

c) Za veću vrijednost λ (nađenu pod b) ispitati linearnu zavisnost vektora: $\overrightarrow{AD}$, $\overrightarrow{BD}$ i $\overrightarrow{AC}$. U slučaju linearne zavisnosti razložiti vektor $\overrightarrow{AC}$ preko $\overrightarrow{AD}$ i $\overrightarrow{BD}$

Rj.



a) $D = ?$

Šta znamo za paralelogram?

Paralelogram ima dva para naspremnih podudarnih stranica, pa:

$\overrightarrow{AD} = \overrightarrow{BC}$ i $\overrightarrow{AB} = \overrightarrow{DC}$

$$\left.\begin{array}{l} A(-3,2,\lambda) \\ D(x,y,z) \end{array}\right\} \Rightarrow \overrightarrow{AD}(x+3, y-2, z-\lambda)$$

$$\left.\begin{array}{l} B(3,-3,1) \\ C(5,\lambda,2) \end{array}\right\} \Rightarrow \overrightarrow{BC}(2, \lambda+3, 1)$$

$$\Rightarrow \begin{array}{l} x+3=2 \\ y-2=\lambda+3 \\ z-\lambda=1 \\ \hline x=-1 \\ y=\lambda+5 \\ z=\lambda+1 \end{array}$$

$D(-1, \lambda+5, \lambda+1)$

II način: posmatramo sredine dijagonala (ostavljamo studentima za vježbu)

b) $\lambda = ?$ $\overrightarrow{AD} = \sqrt{14}$

$$\left.\begin{array}{l} A(-3,2,\lambda) \\ D(-1,\lambda+5,\lambda+1) \end{array}\right\} \Rightarrow \overrightarrow{AD}(2, \lambda+3, 1)$$

$$\begin{array}{l} |\overrightarrow{AD}| = \sqrt{4+(\lambda+3)^2+1} \\ |\overrightarrow{AD}| = \sqrt{14} \\ \hline 4+\lambda^2+6\lambda+9+1 = 14 \end{array}$$

$$\begin{array}{l} \lambda^2+6\lambda=0 \\ \lambda(\lambda+6)=0 \\ \hline \end{array} \qquad \begin{array}{l} \lambda_1 = 0 \\ \lambda_2 = -6 \end{array}$$

Za $\lambda=0$ ili $\lambda=-6$ imamo $\overrightarrow{AD} = \sqrt{14}$.

c) $\lambda = 0$ Rj. $\overrightarrow{AC} = 2\overrightarrow{AD} - \overrightarrow{BD}$

razlaganje vektora $\overrightarrow{AC}$

Skalarni proizvod

(dva vektora)

$$\vec{a}\cdot\vec{b} = |\vec{a}|\cdot|\vec{b}|\cdot\cos\sphericalangle(\vec{a},\vec{b}) \quad \Rightarrow \quad \cos\sphericalangle(\vec{a},\vec{b}) = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|}$$

$\vec{a}(x_1, y_1, z_1)$
$\vec{b}(x_2, y_2, z_2)$

$$\vec{a}\cdot\vec{b} = x_1\cdot x_2 + y_1\cdot y_2 + z_1\cdot z_2$$

za $\vec{a}\cdot\vec{b}=0$ vekori $\vec{a}$ i $\vec{b}$ su okomiti

1. Dati su vektori $\vec{a}=(1,2,1)$ i $\vec{b}=(2,1,-1)$. Izračunati $\vec{a}\cdot\vec{b}$, $(\vec{a}-\vec{b})^2$, $\sqrt{\vec{a}^2}$ i $\sphericalangle(\vec{a},\vec{b})$.

Rj: $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\sphericalangle(\vec{a},\vec{b})$

$\vec{a}\cdot\vec{b} = (1,2,1)(2,1,-1) = 2+2-1 = 3$ $\qquad \vec{a}\cdot\vec{b}=3$

$\vec{a}=(1,2,1)$
$\vec{b}=(2,1,-1)$

$\vec{a}-\vec{b} = (-1,1,2)$

$(\vec{a}-\vec{b})^2 = (-1,1,2)(-1,1,2) = 1+1+4 = 6$ $\qquad (\vec{a}-\vec{b})^2 = 6$

$\vec{a}^2 = (1,2,1)(1,2,1) = 1+4+1 = 6$

$\sqrt{\vec{a}^2} = \sqrt{6}$ $\qquad |\vec{a}| = \sqrt{\vec{a}^2} = \sqrt{6}$, $|\vec{b}| = \sqrt{4+1+1} = \sqrt{6}$

$$\cos\sphericalangle(\vec{a},\vec{b}) = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{3}{\sqrt{6}\cdot\sqrt{6}} = \frac{3}{6} = \frac{1}{2} \Rightarrow \sphericalangle(\vec{a},\vec{b}) = 60^\circ$$

ugao između vektora $\vec{a}$ i $\vec{b}$.

2. Odrediti parametar λ tako da vektori $\vec{a}(2a^\lambda, \lambda, \lambda-1)$ i $\vec{b}(\lambda+1, \lambda-2, 0)$ imaju isti intenzitet a zatim naći ugao između njih.

Rj. $|\vec{a}| = |\vec{b}|$

$$\left.\begin{aligned} |\vec{a}| &= \sqrt{(2a^\lambda)^2+\lambda^2+(\lambda-1)^2} \\ |\vec{b}| &= \sqrt{(\lambda+1)^2+(\lambda-2)^2+0^2} \end{aligned}\right\} \Rightarrow$$

$4a^{2\lambda} + \cancel{\lambda^2} + \cancel{\lambda^2} \underline{-2\lambda} + \underline{\underline{1}} =$

$= \cancel{\lambda^2} \underline{+2\lambda} \underline{\underline{+1}} + \cancel{\lambda^2} \underline{-4\lambda} + 4$

$4a^{2\lambda} = 4$

$a^{2\lambda} = 1$

$$a^{2\lambda} = a^0 \quad \Rightarrow \quad 2\lambda = 0$$
$$\lambda = 0$$

Za $\lambda = 0$ vektori $\vec{a}$ i $\vec{b}$ imaju isti intenzitet.

$$\left.\begin{array}{l} \vec{a}(2, 0, -1) \\ \vec{b}(1, -2, 0) \end{array}\right\} \Rightarrow \vec{a}\cdot\vec{b} = 2 + 0 + 0 = 2$$

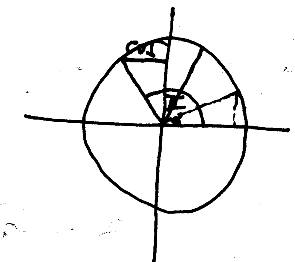
$$\cos\sphericalangle(\vec{a}, \vec{b}) = \frac{2}{\sqrt{5}\cdot\sqrt{5}} = \frac{2}{5}$$

$$\sphericalangle(\vec{a}, \vec{b}) = \arccos\frac{2}{5}$$ ugao između vektora

3. Zadani su vektori $\vec{p} = \lambda\vec{a} + 17\vec{b}$ i $\vec{q} = 3\vec{a} - \vec{b}$ gdje je $|\vec{a}| = 2$, $|\vec{b}| = 5$ a $\sphericalangle(\vec{a}, \vec{b}) = \frac{2\pi}{3}$ (ugao između vektora $\vec{a}$ i $\vec{b}$). Odrediti koeficijent λ tako da vektori $\vec{p}$ i $\vec{q}$ budu međusobno okomiti.

Rj. $\vec{p}\cdot\vec{q} = 0$ (uslov okomitosti)

$$\vec{p}\cdot\vec{q} = (\lambda\vec{a} + 17\vec{b})(3\vec{a} - \vec{b}) = 3\lambda\vec{a}^2 - \lambda\vec{a}\vec{b} + 51\vec{a}\vec{b} - 17\vec{b}^2$$
$$= 3\lambda\vec{a}^2 + (51 - \lambda)\vec{a}\vec{b} - 17\vec{b}^2$$



$$\vec{a}^2 = \vec{a}\cdot\vec{a} = |\vec{a}|\cdot|\vec{a}|\cdot\cos\sphericalangle(\vec{a}, \vec{a}) = 2\cdot 2\cdot\cos 0° = 4$$
$$\vec{a}\cdot\vec{b} = |\vec{a}|\cdot|\vec{b}|\cdot\cos\sphericalangle(\vec{a}, \vec{b}) = 2\cdot 5\cdot\cos\frac{2\pi}{3} = 10\cdot\left(-\sin\frac{\pi}{6}\right) = 10\cdot\left(-\frac{1}{2}\right) = -5$$
$$\vec{b}^2 = \vec{b}\cdot\vec{b} = |\vec{b}|\cdot|\vec{b}|\cdot\cos\sphericalangle(\vec{b}, \vec{b}) = 5\cdot 5\cdot\cos 0° = 25$$

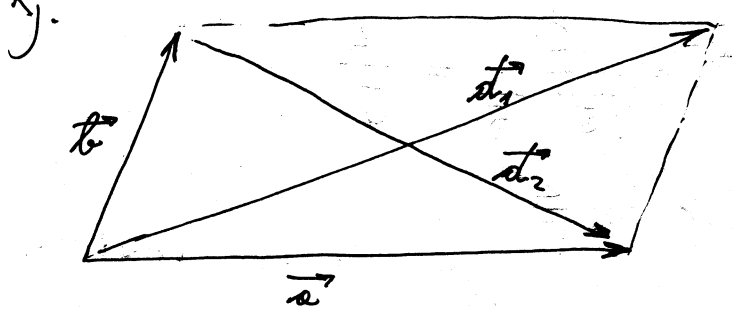
$$\vec{p}\cdot\vec{q} = 0$$
$$3\lambda\vec{a}^2 + (51 - \lambda)\vec{a}\vec{b} - 17\vec{b}^2 = 0$$
$$3\lambda\cdot 4 + (51 - \lambda)\cdot(-5) - 17\cdot 25 = 0$$
$$12\lambda - 225 + 5\lambda - 425 = 0$$
$$17\lambda - 680 = 0$$
$$17\lambda = 680$$

$$\lambda = 40$$

Za $\lambda = 40$ vektori $\vec{p}$ i $\vec{q}$ su međusobno okomiti.

4. Naći dužine dijagonala i ugao između njih, paralelograma konstruisanog nad vektorima $\vec{a} = 2\vec{m} + \vec{n}$ i $\vec{b} = \vec{m} - 2\vec{n}$, gdje su $\vec{m}$ i $\vec{n}$ jedinični vektori koji obrazuju ugao od $\frac{\pi}{3}$.

Rj.



$\vec{a} = 2\vec{m} + \vec{n}$

$\vec{b} = \vec{m} - 2\vec{n}$

$\vec{d_1} = \vec{a} + \vec{b}$

$\vec{d_2} = \vec{a} - \vec{b}$

$|\vec{d_1}| = ? \quad |\vec{d_2}| = ? \quad \sphericalangle(\vec{d_1}, \vec{d_2}) = ?$

$\vec{m}$ i $\vec{n}$ su jedinični vektori $\Rightarrow |\vec{m}| = |\vec{n}| = 1$

$\vec{m} \cdot \vec{n} = |\vec{m}| \cdot |\vec{n}| \cdot \cos\sphericalangle(\vec{m}, \vec{n}) = 1 \cdot 1 \cdot \cos\frac{\pi}{3} = \frac{1}{2}$

$\vec{a} + \vec{b} = 3\vec{m} - \vec{n}$

$|\vec{a} + \vec{b}| = \sqrt{(3\vec{m} - \vec{n})^2} = \sqrt{9\vec{m}^2 - 6\vec{m}\vec{n} + \vec{n}^2} = \sqrt{9 - 3 + 1} = \sqrt{7}$

$\vec{a} - \vec{b} = \vec{m} + 3\vec{n}$

$|\vec{a} - \vec{b}| = \sqrt{(\vec{m} + 3\vec{n})^2} = \sqrt{\vec{m}^2 + 6\vec{m} \cdot \vec{n} + 9\vec{n}^2} = \sqrt{1 + 3 + 1} = \sqrt{13}$

$\vec{d_1} \cdot \vec{d_2} = |\vec{d_1}| \cdot |\vec{d_2}| \cdot \cos\sphericalangle(\vec{d_1}, \vec{d_2})$

$$\cos\sphericalangle(\vec{d_1}, \vec{d_2}) = \frac{\vec{d_1} \cdot \vec{d_2}}{|\vec{d_1}| \cdot |\vec{d_2}|}$$

$\vec{d_1} \cdot \vec{d_2} = (\vec{a} + \vec{b})(\vec{a} - \vec{b}) = \vec{a}^2 - \vec{b}^2 = |\vec{a}|^2 - |\vec{b}|^2$

$|\vec{a}| = \sqrt{(2\vec{m} + \vec{n})^2} = \sqrt{4\vec{m}^2 + 4\vec{m}\vec{n} + \vec{n}^2} = \sqrt{4 + 2 + 1} = \sqrt{7}$

$|\vec{b}| = \sqrt{(\vec{m} - 2\vec{n})^2} = \sqrt{\vec{m}^2 - 4\vec{m}\vec{n} + 4\vec{n}^2} = \sqrt{1 - 2 + 4} = \sqrt{3}$

$\vec{d_1} \cdot \vec{d_2} = 7 - 3 = 4 \qquad \cos\sphericalangle(\vec{d_1}, \vec{d_2}) = \frac{4}{\sqrt{91}}$

Dijagonale $\vec{d_1}$ i $\vec{d_2}$ paralelograma imaju dužine $\sqrt{7}$ i $\sqrt{13}$ a obrazuju ugao od $\arccos\frac{4}{\sqrt{91}}$.

(#) Dati su vektori $\vec{a}=(8-\lambda, 3, -1-\lambda)$, $\vec{b}=(7,1,0)$ i $\vec{c}=(7,7,0)$. Odrediti parametar λ tako da $\sphericalangle(\vec{a},\vec{b})=\sphericalangle(\vec{a},\vec{c})$ (da ugao između vektora $\vec{a}$ i $\vec{b}$ bude jednak uglu između vektora $\vec{a}$ i $\vec{c}$), pa za dobijenu vrijednost λ odrediti veličinu ugla.

Rj.

$$\vec{a}\cdot\vec{b}=|\vec{a}|\cdot|\vec{b}|\cdot\cos\sphericalangle(\vec{a},\vec{b})$$

$$\vec{a}\cdot\vec{b}=(8-\lambda, 3, -1-\lambda)\cdot(7,1,0)=56-7\lambda+3=59-7\lambda$$

$$|\vec{a}|=\sqrt{(8-\lambda)^2+3^2+(-1-\lambda)^2}$$

$$|\vec{b}|=\sqrt{49+1}=\sqrt{50}=5\sqrt{2} \qquad \cos\sphericalangle(\vec{a},\vec{b})=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|}$$

$$|\vec{c}|=\sqrt{49+49}=7\sqrt{2} \qquad \cos\sphericalangle(\vec{a},\vec{c})=\frac{\vec{a}\cdot\vec{c}}{|\vec{a}|\cdot|\vec{c}|}$$

Kako tražimo λ tako da $\sphericalangle(\vec{a},\vec{b})=\sphericalangle(\vec{a},\vec{c}) \Rightarrow$

$$\Rightarrow \cos\sphericalangle(\vec{a},\vec{b})=\cos\sphericalangle(\vec{a},\vec{c}) \Rightarrow \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}=\frac{\vec{a}\cdot\vec{c}}{|\vec{c}|}$$

$$\vec{a}\cdot\vec{c}=(8-\lambda, 3, -1-\lambda)(7,7,0)=56-7\lambda+21=77-7\lambda$$

$$\frac{59-7\lambda}{5\sqrt{2}}=\frac{77-7\lambda}{7\sqrt{2}} \quad /\cdot 35\sqrt{2}$$

$$413-49\lambda=385-35\lambda$$

$$14\lambda=28$$

$$\lambda=2$$

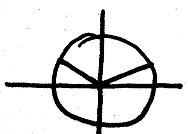
Za vrijednost $\lambda=2$ imamo $\sphericalangle(\vec{a},\vec{b})=\sphericalangle(\vec{a},\vec{c})$

$$\lambda=2 \Rightarrow \vec{a}=(6,3,-3)$$

$$|\vec{a}|=\sqrt{36+9+9}=\sqrt{54}=3\sqrt{6}$$

$$\cos\sphericalangle(\vec{a},\vec{b})=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|}=\frac{(6,3,-3)\cdot(7,1,0)}{3\sqrt{6}\cdot5\sqrt{2}}=\frac{42+3}{15\sqrt{12}}=\frac{45}{15\sqrt{4\cdot3}}=$$

$$=\frac{3}{2\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}}=\frac{\sqrt{3}}{2} \qquad \cos\sphericalangle(\vec{a},\vec{b})=\frac{\sqrt{3}}{2} \Rightarrow$$

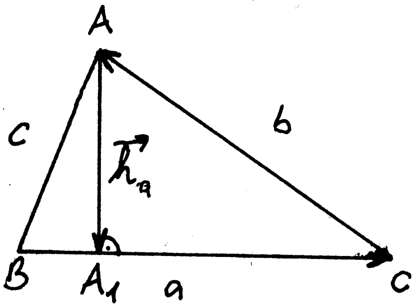


$$\sphericalangle(\vec{a},\vec{b})=\frac{\pi}{6}=30^\circ \text{ ili } \sphericalangle(\vec{a},\vec{b})=\frac{11\pi}{6}=330^\circ$$

veličina ugla

Odrediti vektor visine $\vec{h}_a$ iz vrha A trougla ΔABC ako je $\vec{BC} = \vec{m} + 2\vec{n}$, $\vec{CA} = 2\vec{m} - \vec{n}$, $|\vec{m}| = |\vec{n}| = \sqrt{3}$, $\measuredangle(\vec{m}, \vec{n}) = \frac{\pi}{2}$.

Rj.



$$\vec{AB} = \vec{BC} + \vec{CA} = \vec{m} + 2\vec{n} + 2\vec{m} - \vec{n} = 3\vec{m} + \vec{n}$$

$$\vec{h}_a = ?$$

$$\vec{h}_a = x\vec{m} + y\vec{n}$$

$$\vec{h}_a \perp \vec{BC} \Rightarrow \vec{h}_a \cdot \vec{BC} = 0 \quad \text{tj.}$$

$$(x\vec{m} + y\vec{n}) \cdot (\vec{m} + 2\vec{n}) = x\vec{m}^2 + 2x\vec{m}\cdot\vec{n} + y\vec{m}\vec{n} + y\vec{n}^2 \overset{(*)}{=}$$

$$\vec{m}^2 = |\vec{m}|^2 = 3$$

$$\vec{n}^2 = |\vec{n}|^2 = 3 \qquad \ldots(*)$$

$$\vec{m}\cdot\vec{n} = |\vec{m}||\vec{n}| \cos\measuredangle(\vec{m}, \vec{n}) = 0$$

$$\overset{(*)}{=} 3x + 3y = 0$$

tj. $x + y = 0$

$x = -y$

$$P_{\Delta ABC} = \sqrt{s(s-a)(s-b)(s-c)}$$

$$P_{\Delta ABC} = \frac{|\vec{h}_a| \cdot |\vec{BC}|}{2}$$

$$a^2 = |\vec{BC}|^2 = \vec{BC}^2 = (\vec{m} + 2\vec{n})^2 =$$
$$= \vec{m}^2 + 4\vec{m}\vec{n} + 4\vec{n}^2 = 3 + 4\cdot 3 = 15$$

$$b^2 = |\vec{CA}|^2 = \vec{CA}^2 = (2\vec{m} - \vec{n})^2 = 4\vec{m}^2 - 4\vec{m}\vec{n} + \vec{n}^2$$
$$= 15$$

$$c^2 = |\vec{AB}|^2 = \vec{AB}^2 = (3\vec{m} + \vec{n})^2 = 9\vec{m}^2 + 6\vec{m}\vec{n} + \vec{n}^2 = 30$$

$a = \sqrt{15}$, $b = \sqrt{15}$, $c = \sqrt{30}$

$$s = \frac{a+b+c}{2} = \frac{2\sqrt{15} + \sqrt{30}}{2} = \sqrt{15} + \frac{\sqrt{2}}{2}\sqrt{15} = \sqrt{15} + \frac{\sqrt{30}}{2}$$

$$P_{\Delta ABC} = \sqrt{\left(\sqrt{15} + \frac{\sqrt{2}}{2}\sqrt{15}\right)\left(\frac{\sqrt{2}}{2}\sqrt{15}\right)\left(\frac{\sqrt{2}}{2}\sqrt{15}\right)\left(\sqrt{15} - \frac{\sqrt{30}}{2}\right)} =$$

$$= \sqrt{\left(15 - \frac{30}{4}\right) \cdot \frac{1}{4} \cdot 15} = \sqrt{\frac{30}{4} \cdot \frac{1}{4} \cdot 15} = \frac{15}{4}\sqrt{2} \qquad \ldots(\Delta)$$

$$P_{\Delta ABC} = \frac{|\vec{h}_a| \cdot \sqrt{15}}{2} \overset{(\Delta)}{\Longrightarrow} |\vec{h}_a| \cdot \sqrt{15} = \frac{15}{2}\sqrt{2} \Rightarrow |\vec{h}_a| = \frac{15}{2}\sqrt{\frac{2}{15}}$$

$$|\vec{h}_a|^2 = \frac{15^2}{2^2} \cdot \frac{2}{15} = \frac{15}{2} = \vec{h}_a^2 = (x\vec{m} + y\vec{n})^2 = x^2\vec{m}^2 + 2xy\vec{m}\vec{n} + y^2\vec{n}^2$$

$$= 3x^2 + 3y^2 \quad \text{tj.} \quad 3x^2 + 3y^2 = \frac{15}{2} \Rightarrow x^2 + y^2 = \frac{5}{2} \quad \text{kako } x = -y$$

$$2y^2 = \frac{5}{2} \qquad y_{1,2} = \pm\frac{\sqrt{5}}{2} \qquad y_1 = \frac{\sqrt{5}}{2} \Rightarrow x_1 = -\frac{\sqrt{5}}{2} \qquad \vec{h}_a = \pm\left(\frac{\sqrt{5}}{2}\vec{m} - \frac{\sqrt{5}}{2}\vec{n}\right)$$

$$y^2 = \frac{5}{4} \qquad y_2 = -\frac{\sqrt{5}}{2} \Rightarrow x_2 = \frac{\sqrt{5}}{2}$$

$\pm$ zavisi od $\vec{AA_1}$ ili $\vec{A_1A}$

(#) Dati su vektori $\vec{a}=(\lambda,-\lambda-1,-\lambda-2)$, $\vec{b}=(2,-1,-7)$ i $\vec{c}=(6,-3,-3)$. Odrediti parametar λ tako da $\measuredangle(\vec{a},\vec{b})=\measuredangle(\vec{a},\vec{c})$ (ugao između vektora $\vec{a}$ i $\vec{b}$ bude jednak uglu između vektora $\vec{a}$ i $\vec{c}$), pa za dobijenu vrijednost λ odrediti veličinu ugla.

Rj.

$\vec{a}=(\lambda,-\lambda-1,-\lambda-2)$

$\vec{b}=(2,-1,-7)$

$\vec{c}=(6,-3,-3)$

$$\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\measuredangle(\vec{a},\vec{b})$$

$$\cos\measuredangle(\vec{a},\vec{b})=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$

isto tako

$$\cos\measuredangle(\vec{a},\vec{c})=\frac{\vec{a}\cdot\vec{c}}{|\vec{a}||\vec{c}|}$$

Imamo $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}=\dfrac{\vec{a}\cdot\vec{c}}{|\vec{a}||\vec{c}|} \Rightarrow \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}=\dfrac{\vec{a}\cdot\vec{c}}{|\vec{c}|}$

$$\left.\begin{aligned}
\vec{a}\cdot\vec{b}&=2\lambda+\lambda+1+7\lambda+14=10\lambda+15\\
\vec{a}\cdot\vec{c}&=6\lambda+3\lambda+3+3\lambda+6=12\lambda+9\\
|\vec{b}|&=\sqrt{4+1+49}=\sqrt{54}=\sqrt{6\cdot 9}=3\sqrt{6}\\
|\vec{c}|&=\sqrt{36+9+9}=\sqrt{54}=3\sqrt{6}
\end{aligned}\right\} \Rightarrow \frac{10\lambda+15}{3\sqrt{6}}=\frac{12\lambda+9}{3\sqrt{6}}$$

$$10\lambda-12\lambda=9-15$$

$$2\lambda=6$$

$$\lambda=3$$

tražena vrijednost za λ

$\vec{a}=(3,-4,-5)$

$|\vec{a}|=\sqrt{9+16+25}=\sqrt{50}=\sqrt{2\cdot 25}=5\sqrt{2}$

$|\vec{b}|=3\sqrt{6}$

$\vec{a}\cdot\vec{b}=30+15=45$

$$\cos\measuredangle(\vec{a},\vec{b})=\frac{45}{5\sqrt{2}\cdot 3\sqrt{6}}=\frac{3}{\sqrt{2}\cdot\sqrt{2\cdot 3}}$$

$$\cos\measuredangle(\vec{a},\vec{b})=\frac{3}{2\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}}=\frac{3\sqrt{3}}{2\cdot 3}=\frac{\sqrt{3}}{2}$$

$$\cos\measuredangle(\vec{a},\vec{b})=\frac{\sqrt{3}}{2} \Rightarrow \measuredangle(\vec{a},\vec{b})=30^\circ$$

veličina ugla između vektora

(Zadaci su skinuti sa stranice: \pf.unze.ba\nabokov
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